To facilitate the steps that follow, it is convenient to introduce the function $\Phi(r, t)$ defined by the relation

$$\Phi(r, t) = \int_{0-}^{t} R(\xi - \xi') \frac{\partial P}{\partial \tau} d\tau \qquad (40)$$

In terms of this function, (39) assumes the simpler form

$$\frac{\partial \sigma_r}{\partial r} = \Phi(r, t) + \frac{2}{r^3} \int_{0-}^{t} R(\xi - \xi') \frac{dc}{d\tau} d\tau \qquad (41)$$

Upon integration of (41) and as a consequence of the boundary condition $\sigma_r(r_1, t) = 0$, one obtains an explicit relation for $\sigma_r$:

$$\sigma_r = \int_{r_1}^{r} \Phi(p, t)\, d\rho + 2 \int_{r_1}^{r} \frac{1}{\rho^3} \int_{0-}^{t} \times R(\xi - \xi') \frac{dc}{d\tau} d\tau\, d\rho \qquad (42)$$

The unknown function $c(t)$ will be determined from the condition of continuity of radial stress and displacement at the cylinder-shell interface, which yields the relation

$$\epsilon_\theta(r_2, t) = (1 + \nu_s)\alpha_s\Theta_s - \sigma_r(r_2, t) \frac{r_2}{h} \frac{1 - \nu_s^2}{E_s} \qquad (43)$$

where the suffix $s$ pertains to the shell and $\Theta_s$ is the average temperature over the shell thickness. In particular, as a direct consequence of (42),

$$\sigma_r(r_2, t) = \int_{r_1}^{r_2} \Phi d\rho + 2 \int_{0-}^{t} F(t, \tau) \frac{dc}{d\tau} d\tau \qquad (44)$$

where, after reversal of the order of integration on the right-hand side of (42),

$$F(t,\tau) = \int_{r_1}^{r_2} \frac{1}{r^3} R(\xi - \xi')dr \qquad (45)$$

Also, from (25),

$$\epsilon_\theta(r_2, t) = \mu\sigma_r(r_2, t) + [(ct)/r_2^2] \qquad (46)$$

From (43, 44, and 46), one obtains the following integral equation in terms of $c(t)$:

$$r_2^2(1 + \nu_s)\alpha_s\Theta_s - r_2^2 \lambda \int_{r_1}^{r_2} \Phi(r, t)dr = c(t) + r_2^2 \lambda \int_{0-}^{t} F(t, \tau) \frac{dc}{d\tau} d\tau \qquad (47)$$

where

$$\lambda = \frac{1 + \nu}{3K} + \frac{1 - \nu_s^2}{E_s}\left(\frac{r_2}{h}\right)$$

Equation (47) is, essentially, a Volterra integral equation of the second kind, from which $c(t)$ may be determined. Upper and lower bounds to the solution of this equation with the type of Kernal occurring here have been established.[3] Substitution of $c(t)$ in (42) solves the problem completely. The hoop stress may be found immediately from the relation

$$\sigma_\theta = \sigma_r + r\,(r\sigma_r/\partial r) \qquad (48)$$

Also, other boundary conditions such as $\epsilon_\theta(r_2, t) = 0$ (rigid shell) or $\sigma_r(r_2, t) = 0$ (free surface) are particular cases of the situation dealt with previously.

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# Sudden Expansion of a Bounded Jet at a High Pressure Ratio

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The free jet from a convergent nozzle at a high pressure ratio expands through a series of expansion waves and is then theoretically refocused to a second node as the expansion waves are reflected as compression waves from a constant pressure boundary. The characteristics of such an underexpanded jet in free air are treated in detail in Ref. 1. The outstanding characteristics of the free jet are that the jet diameter is larger than the nozzle exit diameter, that there is a supersonic area in the center of the jet at a static pressure less than the jet boundary pressure, and that the jet is periodic. If a duct is placed around the jet, the duct has no influence unless the duct diameter is less than the maximum jet diameter, since the constant pressure boundary is maintained by pressure propagation in the space between the duct wall and the jet boundary. However, if the duct wall intersects the jet boundary, the boundary condition is altered so that the expansion waves emanating from the nozzle exit plane are reflected from the solid boundary as expansion waves, and the flow is thus straightened to flow parallel to the duct walls. This results in a nonuniform supersonic flow field. Viscous effects then tend to dissipate the waves and to establish a uniform flow field.

To ascertain that boundary-layer effects would not predominate and would allow the expected flow field to form, a series of tests was conducted over a limited range of test conditions. High pressure air was passed through a nozzle mounted in a cylindrical duct whose area was approximately four times the nozzle area. The flow characteristics were observed by measuring the static pressure along the centerline of the duct-nozzle combination. For comparison, tests were conducted at the same conditions with a convergent-divergent nozzle in place of the convergent nozzle.

As illustrated in Fig. 1, the centerline pressure immediately behind the exit plane of the convergent nozzle drops to a very low value in the same manner that the centerline pressure drops in a free jet. This is followed by a compression to the same pressure level that is obtained by a convergent-divergent nozzle of the same area ratio. This illustrates that, at several diameters from the exit plane of a sudden expansion, the flow field, pressure, and Mach number would be similar to that obtained by a smooth expansion in a convergent-divergent

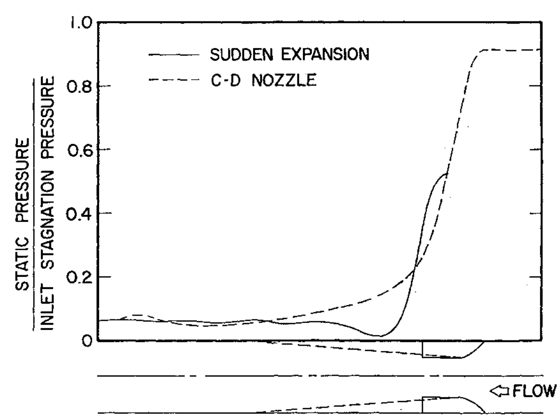


**Fig. 1 Centerline pressure of a sudden expansion of a bounded jet.**

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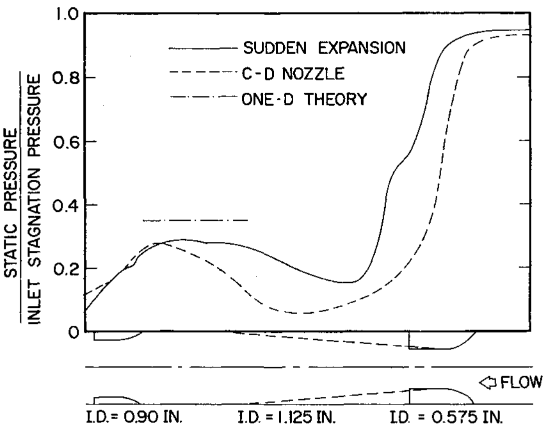


**Fig. 2 Comparison of a sudden expansion and a convergent-divergent nozzle with a medium back pressure.**

nozzle. This corresponds to the flow field that one computes by applying the method of characteristics to the sudden expansion.

The resemblance of the flow behind a sudden expansion to a convergent-divergent nozzle is further illustrated in Figs. 2 and 3. To increase the back pressure, a second nozzle was placed approximately 8 diam downstream from the primary nozzle. Simple one-dimensional theory postulates that, to maintain choked flow in both nozzles, the product of nozzle area and stagnation pressure must remain a constant. Both the convergent-divergent nozzle and the sudden expansion approximate this simple rule as illustrated by the measured centerline pressures taken far downstream from the exit plane. The convergent-divergent nozzle follows the theoretically predicted characteristic. In the divergent portion, the flow is initially supersonic, it passes through a series of normal shock waves which take the place of a single normal shock because of boundary-layer interaction,[2] and the shock waves are followed by a subsonic diffusion. The strength and position of the normal shock are determined by the stagnation pressure loss necessary to match the choked flow conditions at the downstream throat. The flow field behind the sudden expansion achieves the same pressure level prior to the second nozzle. Near the exit plane, the expansion and recompression by expansion and compression waves are evident. The expansion characteristics of the wave pattern are determined by the back pressure so that the over-all stagnation pressure loss is identical to the stagnation pressure loss that is achieved by the normal shock wave pattern in the convergent-divergent nozzle.

The second nozzle must be located downstream well beyond the first expansion wave and wall reflection pattern. If the second nozzle is moved closer, for example, within 2 to 4

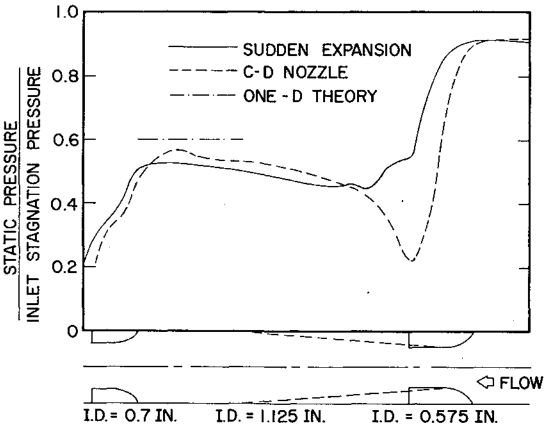


**Fig. 3 Comparison of a sudden expansion and a convergent-divergent nozzle with a high back pressure.**

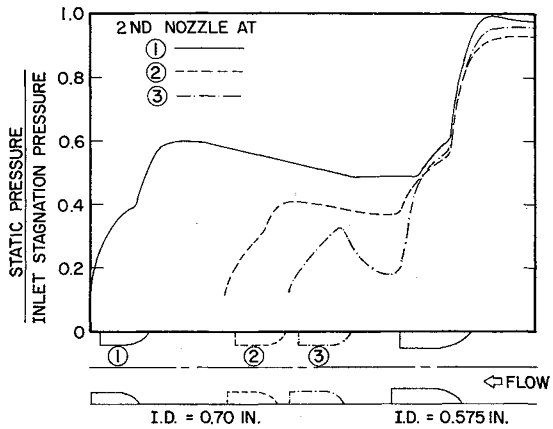


**Fig. 4 Effect of the location of the second nozzle.**

nozzle diam, then the second nozzle interferes with the primary expansion wave pattern and produces a significantly different flow field. This phenomena is illustrated in Fig. 4 where the centerline pressure is plotted for tests with the second nozzle at different locations relative to the exit plane of the first nozzle.

These experiments confirm that the sudden expansion of a bounded jet achieves approximately the same pressure level and, therefore, Mach number, as if the jet were expanded in a convergent-divergent nozzle. A longitudinal distance of 8 nozzle diam is sufficient to establish a relatively uniform flow field. The flow field immediately behind the sudden expansion from the nozzle diameter to the duct diameter resembles the flow field immediately behind an underexpanded nozzle in free air.

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# Condensation Studies in Hotshot Tunnels


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### Introduction

A NUMBER of studies has been made to determine the influence of air condensation on the stream-pressure characteristics and to establish the degree of supersaturation which may be obtained. The most recent work, reported by Daum[1] and Dayman,[2] extends the earlier studies into the


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